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Asymptotic Behavior in Polarized and Half-Polarized $U(1)$ Symmetric Vacuum Spacetimes

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Abstract

We use the Fuchsian algorithm to study the behavior near the singularity of certain families of $U(1)$ Symmetric solutions of the vacuum Einstein equations (with the $U(1)$ isometry group acting spatially). We consider an analytic family of polarized solutions with the maximum number of arbitrary functions consistent with the polarization condition (one of the “gravitational degrees of freedom” is turned off) and show that all members of this family are asymptotically velocity term dominated (AVTD) as one approaches the singularity. We show that the same AVTD behavior holds for a family of “half polarized” solutions, which is defined by adding one extra arbitrary function to those characterizing the polarized solutions. (The full set of non-polarized solutions involves two extra arbitrary functions). We begin to address the issue of whether AVTD behavior is independent of the choice of time foliation by showing that indeed AVTD behavior is seen for a wide class of choices of harmonic time in the polarized and half-polarized ($U(1)$ Symmetric vacuum) solutions discussed here.

1 Introduction

During the last few years, our understanding of the behavior of cosmological solutions¹ near their big bang singularities has increased significantly. There is more and more evidence, both numerical [2] [3] [4] [5] [6] [7] and analytical [8] [9] [10] [11], that wider and wider families of such spacetimes exhibit either asymptotically velocity term dominated (“AVTD”) behavior, or Mixmaster behavior, in a neighborhood of their singularities.

A solution has AVTD behavior if the metric (when expressed in suitable gauge) asymptotically behaves at each spatial point like a Kasner spacetime metric, with the Kasner parameters generally varying from point to point². A solution exhibits Mixmaster behavior if, instead of asymptotic Kasner evolution at each point, one asymptotically has at each point the oscillatory behavior of bouncing from one Kasner epoch to another, as seen in Bianchi IX cosmologies. The idea that a generic solution should have one or the other of these behaviors was first suggested by Belinskii, Khalatnikov, and Lifschitz [12, 13], and is therefore known as the “BKL conjecture.”

For spacetimes with a $U(1)$ isometry group and T^3 spatial topology, there is strong numerical support for the BKL conjecture. Extensive numerical studies of these spacetimes indicate that if the Killing field generating the isometry is hypersurface orthogonal (these are the so-called “polarized solutions,” since one of the two gravitational degrees of freedom is essentially turned off), then AVTD behavior is seen [6]; while for generic $U(1)$ Symmetric spacetimes with T^3 spatial topology, one finds Mixmaster behavior [7]. These results agree with the conclusions of earlier studies of $U(1)$ Symmetric spacetimes [14], which used formal expansions and heuristic analysis to argue that the BKL conjecture likely holds for them.

In this work, we rigorously prove some of these results, using Fuchsian methods. More specifically, we show that for rather general sets of analytic data for polarized $U(1)$ Symmetric solutions, the solutions are AVTD. Furthermore, we identify a class of $U(1)$ Symmetric solutions, intermediate between the polarized and the generic classes, which is AVTD as well. The solutions in this class, which we call “half-polarized,” are intermediate in the sense that, modulo gauge freedom and the constraint equations, the

¹Here we follow Bartnik’s convention [1] of calling a spacetime a “cosmological solution” if, in addition to being a solution of Einstein’s equations, it is globally hyperbolic, with compact Cauchy surfaces.

²See Section 3 for a more precise definition of AVTD behavior.

polarized solutions are characterized by two arbitrary functions on T^2 , the half-polarized by three such functions, and the nonpolarized by four. So in a rough sense, what we find is a distinct way to turn on half of the second degree of freedom at each point, without removing the AVTD behavior. We further show that the class of $U(1)$ Symmetric solutions with AVTD behavior includes spacetimes which are neither polarized nor half-polarized, since we can find such solutions by applying $SL(2, R)$ parametrized Geroch transformations to the half-polarized solutions and still retain AVTD behavior.

Fuchsian methods have become an important tool in recent years for proving the existence of AVTD behavior in cosmological solutions of Einstein's equations. Kichenassamy and Rendall first used these techniques, in 1998, to establish AVTD behavior in (unpolarized) Gowdy spacetimes [8]. Subsequent work [9] showed that in polarized T^2 Symmetric solutions with nonvanishing twist (the vanishing of the twist characterizes the Gowdy solutions as a subfamily of the T^2 Symmetric cosmological spacetimes), AVTD behavior is found. Especially interesting is the very recent result [11] which uses Fuchsian methods to show that AVTD behavior occurs in cosmological solutions (spatially T^3) of the Einstein-scalar field equations with *no* assumed symmetries.

In none of these works, including the work discussed here, do the Fuchsian methods show that *all* solutions in the family under consideration (general Gowdy, polarized T^2 Symmetric, polarized and half-polarized $U(1)$ Symmetric, etc) necessarily have AVTD behavior. Rather, the Fuchsian methods show that in each family of solutions, there is an analytic subfamily which is defined by the same number of free functions as the full family, and whose members all exhibit AVTD behavior. Thus, in a rough sense, Fuchsian methods show that AVTD behavior is stable, occurring in “open subsets” of the given full family. It is expected, based on numerical evidence, that AVTD behavior is generic in these families; however, this has not been proven yet in any family except for the polarized Gowdy solutions.

The polarized and half-polarized $U(1)$ Symmetric spacetimes with which we work here comprise the least restricted family of vacuum solutions in which AVTD behavior has been proven to exist. It is useful to note that the recent work showing that AVTD behavior occurs in Einstein-scalar field solutions with no symmetry does not imply our $U(1)$ results. Indeed heuristic arguments first presented by Belinskii, Khalatnikov, and Lifschitz [13] indicate that the presence of a nonvanishing scalar field is likely to remove the inevitability of the “potential bounces” which lead to Mixmaster behavior,

with AVTD behavior resulting. It is expected that vacuum solutions with no isometries are generically not AVTD.

It would, of course, be very nice if Fuchsian type methods could be used to rigorously prove that Mixmaster behavior occurs in those families of solutions (like the magnetic Gowdy spacetimes [4], the nonpolarized T^2 Symmetric spacetimes [5], and nonpolarized $U(1)$ Symmetric spacetimes [7]) in which Mixmaster behavior has been observed numerically. However, it is not known how to do this. Indeed, only very recently has Mixmaster behavior been rigorously verified in spatially homogeneous spacetimes [15] [16].

In all Fuchsian studies of cosmological solutions prior to ours, the choice of spacetime foliation has been more or less rigidly determined by the family of spacetimes and the analysis. For example, for the Gowdy and for the T^2 Symmetric spacetimes, the analysis of solutions is simplest if one uses the “areal” (or “Gowdy”) foliation [17] [18], in which the $t = \text{constant}$ hypersurfaces each consist of T^2 orbits of constant area; this foliation is (up to scaling) unique. In our present study of polarized $U(1)$ Symmetric spacetimes, however, the analysis is carried out using a harmonic time foliation, as detailed below. Harmonic time is not unique, and so we have the opportunity here to consider the issue of whether, and in what sense, observed AVTD behavior depends on the choice of foliation. We show that if, in a fixed $U(1)$ Symmetric solution $(T^3 \times R, g)$, AVTD behavior is seen using one choice of harmonic time, then there is a full (two free functions on T^2) family of other choices of harmonic time for $(T^3 \times R, g)$ such that AVTD behavior is seen using each of them as well. We also discuss the extent to which we expect AVTD behavior to be seen using other foliations.

The bulk of this paper is devoted to the verification, using Fuchsian methods, that AVTD behavior occurs in polarized and half-polarized $U(1)$ Symmetric vacuum solutions on $T^3 \times R$. We begin by reviewing (in Section 2), the form of the metric and the form of the field equations we shall work with here. We obtain these forms by starting with the generic metric for $U(1)$ Symmetric spacetimes, and then imposing certain gauge conditions, and certain restrictions on the fields which are preserved by the field equations. We write the field equations in canonical, Hamiltonian form. We note in Section 2 that in setting one of the fields together with its conjugate momentum to zero, we define the polarized solutions. We further note that the half-polarized solutions cannot be characterized by restrictions on the fields at an arbitrary finite time; rather, they are characterized by certain asymptotic conditions near the singularity.

In Section 3, we write the asymptotic (near the singularity) ansatz for the polarized solutions, and then for the half-polarized solutions. The asymptotic ansatz for the polarized fields (forgetting the constraints) involves eight free functions (the “asymptotic data”) on T^2 together with eight remainder functions depending on time as well as on T^2 . (There are eight rather than two because the constraint equations have not been imposed, and some gauge freedom remains.) Substituting this ansatz into the Einstein evolution equations, one obtains evolution equations for these remainder functions (with the free functions on T^2 appearing as parameters). We show in Section 3 that for all choices of the asymptotic data which satisfy certain linear inequalities, these evolution equations take the special Fuchsian form. It then follows from a result of Kichenassamy and Rendall [8], that for any such choice of the asymptotic data, there is a unique solution for the remainder functions, decaying to zero near the singularity. This behavior guarantees that solutions in the (polarized) ansatz form are AVTD. The analysis for solutions in the half-polarized asymptotic ansatz form is similar, with similar conclusions.

The discussion in Section 3 ignores the constraint equations. In Section 4, we show that if the asymptotic data in the asymptotic ansatz expressions satisfy certain asymptotic constraints (67) – (69), then the corresponding solution satisfies the Einstein constraints. The converse is true as well; so the Einstein constraints are (at least among solutions of the evolution equations) equivalent to these asymptotic constraints (67) – (69). It follows that the appropriate sets of free functions, restricted to satisfy the asymptotic constraints, parameterize the corresponding families of AVTD polarized and half-polarized solutions.

The work done in Sections 2-4 presumes a fixed choice of harmonic time foliation. In Section 5, we discuss the relation between alternative harmonic time foliations, and we show that one sees AVTD behavior using any one of them. The asymptotic relation between alternative harmonic time foliations explains why this makes sense, and hints at criteria for predicting whether one sees AVTD behavior using other choices of time foliation and observers.

We show in Section 6, using the $SL(2, R)$ Geroch transformation [19], that there are $U(1)$ Symmetric solutions, beyond those discussed here, in which AVTD behavior is seen. We speculate on further classes of solutions which exhibit AVTD behavior, and make concluding remarks, in Section 7.

2 $U(1)$ Symmetric Vacuum Spacetimes

If we choose 2π -periodic coordinates $(x^a, x^3) = (x^1, x^2, x^3)$ on T^3 , with the Killing field expressed as $\frac{\partial}{\partial x^3}$, then the generic $U(1)$ Symmetric spacetime metric on $T^3 \times R^1$ may be written in the form

$$g = e^{-2\phi} [-N^2 e^{-4\tau} d\tau^2 + e^{-2\tau} e^\Lambda e_{ab} dx^a dx^b] + e^{2\phi} (dx^3 + \beta_a dx^a)^2 \quad , \quad (1)$$

with the functions ϕ , N , and Λ , and the form field β_a , and the unit determinant symmetric tensor field e_{ab} , all functions of x^a and the time coordinate τ (independent of x^3). Here the latin indices a, b take the values 1, 2; note that we shall find it convenient to write $(x^1, x^2) = (u, v)$.

The metric form (1) does involve some restrictions on the gauge freedom, in that we have aligned the Killing field with x^3 for all time, and we have set the corresponding shift vector to zero. We now further restrict the gauge freedom by setting the lapse function N equal to e^Λ . This choice of the lapse does not restrict the initial choice of Cauchy slice; however, once a first slice is chosen, all others are determined. We note that it follows from the condition $N = e^\Lambda$ that the time function τ satisfies the equation

$$\square\tau = 0 \quad (2)$$

where \square is the wave operator corresponding to the metric g .³ Hence this choice of slicing is sometimes called a “harmonic time.”⁴

Before writing down the full system of Einstein’s equations for the $U(1)$ symmetric spacetimes in terms of the metric (1), we wish to make a small restriction which considerably simplifies the analysis. To state this condition, we first write out the projection of the Einstein supermomentum constraint along the Killing field $\frac{\partial}{\partial x^3}$; it takes the form⁵

$$f^a{}_{,a} = 0 \quad (3)$$

³One has $\square\tau = 0$ for \square corresponding to the $2 + 1$ metric $\tilde{g} = -N^2 e^{-4\tau} d\tau^2 + e^{-2\tau} e^\Lambda e_{ab} dx^a dx^b$ as well as for \square corresponding to the $3 + 1$ metric in equation (1).

⁴This terminology is a consequence of the somewhat loose practice of using the word “harmonic” to refer to solutions of the equation $\square\tau = 0$ as well as to the equation $\nabla^2\tau = 0$.

⁵The form of this constraint is reminiscent of the Gauss law constraint for electromagnetism. This reflects the fact that one may view $3 + 1$ vacuum gravity with $U(1)$ symmetry as a Kaluza-Klein theory for $2 + 1$ Einstein-Maxwell fields, plus a Jordan scalar field.

where f^a is the momentum conjugate to β_a . The general solution to (3) can be written as

$$f^a = \epsilon^{ab} w_{,b} + h^a \quad (4)$$

where w is a scalar function, $\epsilon^{ab} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and h^a is dual to a harmonic one-form on T^2 . For the $T^3 \times R$ spacetimes of interest here⁶ one finds (using the rest of Einstein's equations), that $(h^a) = \begin{pmatrix} C^1 \\ C^2 \end{pmatrix}$, for C^1 and C^2 constant in time as well as in space.

So far, no restrictive assumptions have been made. Now, however, we set both C^1 and C^2 to zero. Correspondingly, we write β_a in terms of an integral involving one free function, which we call r (see [20] [6] for details), and it follows that the canonical pair (β_a, f^a) is replaced by the pair (w, r) . These restrictions are consistent with Einstein's equations; they are somewhat analogous to the vanishing of the two "twist constants" which defines the Gowdy spacetimes as a subfamily of the T^2 Symmetric spacetimes. One might wish elsewhere to consider what happens if C^1 or C^2 is nonzero, but we shall not pursue the issue here.

If we now choose the following convenient (and *non*-restrictive) form for the unit determinant tensor e_{ab} ,

$$e_{ab} = \frac{1}{2} \begin{pmatrix} e^{2z} + e^{-2z}(1+x)^2 & e^{2z} + e^{-2z}(x^2 - 1) \\ e^{2z} + e^{-2z}(x^2 - 1) & e^{2z} + e^{-2z}(1-x)^2 \end{pmatrix} \quad (5)$$

(where z and x are functions of u, v and τ), then the Einstein evolution equations for these spacetimes can be expressed as a Hamiltonian system for the fields $\{\phi, \Lambda, w, x, z\}$ and their respective conjugate momenta $\{p, p_\Lambda r, p_x, p_z\}$. The Hamiltonian for this system is

$$H = \int_{T^2} h \, du dv \quad (6)$$

with

$$h = \frac{1}{8} p_z^2 + \frac{1}{2} e^{4z} p_x^2 + \frac{1}{8} p^2 + \frac{1}{2} e^{4\phi} r^2 - \frac{1}{2} p_\Lambda^2 + 2p_\Lambda$$

⁶We note that if one studies $U(1)$ Symmetric spacetimes on $(\Sigma^2 \times S^1) \times R$, for more general surfaces Σ^2 , the same equation (3) appears, and one has the same sort of solution (4). However, for $\Sigma^2 = S^2$, (h^a) necessarily vanishes, while for higher genus Σ^2 , (h^a) can be more complicated.

$$\begin{aligned}
& +e^{-2\tau} \left\{ (e^\Lambda e^{ab})_{,ab} - (e^\Lambda e^{ab})_{,a} \Lambda_{,b} + e^\Lambda [(e^{-2z})_{,u} x_{,v} - (e^{-2z})_{,v} x_{,u}] \right. \\
& \left. + 2e^\Lambda e^{ab} \phi_{,a} \phi_{,b} + \frac{1}{2} e^\Lambda e^{-4\phi} e^{ab} w_{,a} w_{,b} \right\}
\end{aligned} \tag{7}$$

where e^{ab} is the matrix inverse of e_{ab} (see equation (5)). The evolution equations for $\{\phi, \Lambda, w, x, z; p, p_\Lambda, r, p_x, p_z\}$ are obtained via Hamilton's equations, using this function H .

There are also constraint equations which this data must satisfy. They take the form

$$\begin{aligned}
0 &= \mathcal{H}_0 \\
&= h - 2p_\Lambda
\end{aligned} \tag{8}$$

$$\begin{aligned}
0 &= \mathcal{H}_u \\
&= p_z z_{,u} + p_x x_{,u} + p_\Lambda \Lambda_{,u} - p_{\Lambda,u} + p\phi_{,u} + r w_{,u} \\
&\quad + \frac{1}{2} \{ [e^{4z} - (1+x)^2] p_x - (1+x)p_z \}_{,v} \\
&\quad - \frac{1}{2} \{ [e^{4z} + (1-x^2)] p_x - x p_z \}_{,u}
\end{aligned} \tag{9}$$

$$\begin{aligned}
0 &= \mathcal{H}_v \\
&= p_z z_{,v} + p_x x_{,v} + p_\Lambda \Lambda_{,v} - p_{\Lambda,v} + p\phi_{,v} + r w_{,v} \\
&\quad - \frac{1}{2} \{ [e^{4z} - (1-x)^2] p_x + (1-x)p_z \}_{,u} \\
&\quad + \frac{1}{2} \{ [e^{4z} + (1-x^2)] p_x - x p_z \}_{,v}
\end{aligned} \tag{10}$$

The first of these is the superHamiltonian constraint. The others are supermomentum constraints. Recall that one of the supermomentum constraints—equation (3)—has been solved, and is therefore of no further interest. For later purposes, it is important to note that \mathcal{H}_0 is written as a double density, while \mathcal{H}_u and \mathcal{H}_v are expressed as single densities. This fact is irrelevant in seeking solutions to the constraints, but it is important in verifying the preservation of the constraints under evolution (see Section 4).

In an appropriate sense, the Hamiltonian system (6)-(7) with constraints (8)-(10) (and with the remaining gauge freedom described above) has two gravitational degrees of freedom. If one can set one of the functions ϕ or w ,

together with the corresponding conjugate momentum p or r , to zero on some Cauchy surface, and if the evolution equations preserve the vanishing of these two conjugate variables, then one has reduced the system to one gravitational degree of freedom, and the resulting family of solutions is said to be “polarized.” Inspecting the Hamiltonian (6)-(7), one finds that the only conjugate pair for which this can be done is (w, r) . Thus, the polarized $U(1)$ Symmetric spacetimes are those with the canonical variables $\{\phi, \Lambda, x, z; p, p_\Lambda, p_x, p_z\}$, with the Hamiltonian given by (6)-(7) with $w = 0$ and $r = 0$, and with the constraints given by (8)-(10) with $w = 0$ and $r = 0$. We note that for these polarized solutions, the spacetime metric takes the form (1), with $\beta_a = 0$.

As noted in the introduction, we find AVTD behavior in a family of “half-polarized” $U(1)$ Symmetric solutions, as well as in the family of polarized $U(1)$ Symmetric solutions as just described. For these half-polarized spacetimes, rather than setting a conjugate pair of variables to zero, one ties the conjugate pair together, so that effectively the pair involves one free function instead of two. It is not known whether this can be done consistently in terms of initial data on a Cauchy surface. It can, however, be done in terms of asymptotic data, as we show in the next section.

3 Fuchsian Study of the Evolution Equations

While AVTD behavior can be described in terms of asymptotic approach to Kasner evolution of the metric fields at each spatial point (see Introduction), for the purposes of proof, the following formulation is more useful.

Definition (AVTD Behavior)

A cosmological solution (M, g) exhibits AVTD behavior if there exists a global spacelike foliation Σ_t of (M, g) , and there exists a solution (M, \hat{g}) of the Einstein VTD equations (which are obtained by dropping all spatial derivatives in the evolution equations written with respect to Σ_t , and dropping all spatial derivatives in the super Hamiltonian constraint) such that if $t_s \in [-\infty, +\infty]$ labels the singularity in (M, g) , then⁷ $\lim_{t \rightarrow t_s} |g(t) - \hat{g}(t)| = 0$.

⁷Although this definition in principle makes sense for any choice of norm $||$, for our purposes here, we presume that $||$ is the absolute value norm for each of the components of g , evaluated independently for each spatial point, so the convergence is pointwise, L^∞ .

Thus, to show that the spacetimes in a given family of solutions have AVTD behavior, one first needs to find an appropriate foliation, with the singularity occurring at a well-defined value of the time parameter. While it is often useful in Fuchsian analyses to choose coordinates so that the singularity occurs at $t = 0$, for the $U(1)$ Symmetric spacetimes, it is convenient to work with the time coordinate τ (introduced in Section 2), in terms of which the singularity occurs at $\tau \rightarrow \infty$. Note that in terms of $t = e^{-\tau}$, the singularity occurs at $t = 0$; however the use of τ instead of t simplifies the analysis here.

We next identify the VTD equations for the $U(1)$ Symmetric spacetimes. The VTD evolution equations⁸, written with respect to τ and the coordinates chosen in Section 2, can be obtained by varying the Hamiltonian

$$\hat{H} = \int_{T^2} \left[\frac{1}{8} p_z^2 + \frac{1}{2} e^{4z} p_x^2 + \frac{1}{8} p^2 + \frac{1}{2} e^{4\phi} r^2 - \frac{1}{2} p_\Lambda^2 + 2p_\Lambda \right] dudv \quad . \quad (11)$$

As discussed in Section 4, the general solution to the VTD evolution equations can be written out explicitly. In fact, for the purpose of showing that the polarized and the half-polarized $U(1)$ Symmetric solutions have AVTD behavior, we only need a large τ expression for the VTD solution. For the polarized case ($r = 0, w = 0$), we have the following:

Large τ VTD Solution (Polarized)

$$\hat{\phi}(u, v, \tau) = \overset{\circ}{\phi}(u, v) - v_\phi(u, v)\tau \quad (12)$$

$$\hat{p}(u, v, \tau) = -4v_\phi(u, v) \quad (13)$$

$$\hat{\Lambda}(u, v, \tau) = \overset{\circ}{\Lambda}(u, v) + 2\tau - v_\Lambda(u, v)\tau \quad (14)$$

$$\hat{p}_\Lambda(u, v, \tau) = v_\Lambda(u, v) \quad (15)$$

$$\hat{x}(u, v, \tau) = \overset{\circ}{x}(u, v) \quad (16)$$

⁸In this section, we ignore the constraint equations, both for the VTD system and for the full Einstein system.

$$\hat{p}_x(u, v, \tau) = v_x(u, v) \quad (17)$$

$$\hat{z}(u, v, \tau) = \overset{\circ}{z}(u, v) - v_z(u, v)\tau \quad (18)$$

$$\hat{p}_z(u, v, \tau) = -4v_z(u, v) \quad (19)$$

Here v_ϕ and v_z are strictly positive ⁹, and the notation $\hat{\phi}, \hat{p}$, etc, is used to indicate that these are VTD solutions, not Einstein solutions. The eight functions $\{\overset{\circ}{\phi}, v_\phi, \overset{\circ}{\Lambda}, v_\Lambda, \overset{\circ}{x}, v_x, \overset{\circ}{z}, v_z\}$ are all – until we consider the VTD constraints, in Section 4 – treated as free functions on T^2 ; they parameterize the set of solutions of the large τ VTD equations corresponding to \hat{H} in equation (11)

We now write the polarized $U(1)$ Symmetric fields each as the sum of the appropriate term from the large τ VTD solution, plus a remainder term. Specifically, for $\epsilon 0$, we have

Expansion Ansatz (Polarized)

$$\phi = \hat{\phi} + \delta\phi \quad (20)$$

$$p = \hat{p} + e^{-\epsilon\tau}\delta p \quad (21)$$

$$\Lambda = \hat{\Lambda} + \delta\Lambda \quad (22)$$

$$p_\Lambda = \hat{p}_\Lambda + e^{-\epsilon\tau}\delta p_\Lambda \quad (23)$$

$$x = \hat{x} + \delta x \quad (24)$$

$$p_x = \hat{p}_x + e^{-\epsilon\tau}\delta p_x \quad (25)$$

$$z = \hat{z} + \delta z \quad (26)$$

⁹This parametrization of the large τ VTD solution requires this sign condition.

$$p_z = \hat{p}_z + e^{-\epsilon\tau} \delta p_z \quad (27)$$

The aim now is to substitute this ansatz into the Einstein evolution equations, and show that the set of remainder functions $\{\delta\phi, \delta p, \dots, \delta p_z\}$, collectively, satisfies an evolution equation in Fuchsian form. That is, if one uses Ψ to denote a vector whose components consist of $\{\delta\phi, \delta p, \dots, \delta p_z\}$ and certain of their spatial derivatives, then one wants to show that Ψ satisfies the Fuchsian equation¹⁰ [21] [8]

$$\partial_\tau \Psi - A\Psi = e^{-\mu\tau} F(\tau, u, v, \Psi, \nabla\Psi) \quad (28)$$

where A is a matrix which is independent of τ , and for which $\sigma^A[:= \exp(A \ell n \sigma)]$ is uniformly bounded for $0 < \sigma < 1$; where μ is a strictly positive constant; and where F is continuous in τ , is analytic in u, v and Ψ , and satisfies an estimate of the form

$$|F(\tau, u, v, \Psi, \nabla\Psi) - F(\tau, u, v, \Theta, \nabla\Theta)| \leq C[|\Psi - \Theta| + |\nabla\Psi - \nabla\Theta|] \quad , \quad (29)$$

for some constant C , provided that $\Theta, \nabla\Theta, \Psi$, and $\nabla\Psi$ are bounded. (Here “ $||$ ” refers to the pointwise absolute value norm, summed over the relevant components of F, Ψ , etc.)

For our present purposes, one readily verifies that it is sufficient to show that each of the functions $\delta\psi, \delta p$, etc (generically labeled “ q ”) satisfies an evolution equation of the form

$$\partial_\tau q - \nu q = \sum_k e^{-\mu_k\tau} f_k(\tau, u, v, \Psi, \nabla\Psi) \quad (30)$$

where the sum is finite; where $\nu(u, v) \geq 0$; where the $\mu_k(u, v)$ are τ -independent functions, each bounded from below by a strictly positive constant; and where the functions f_k are analytic in u, v, τ, Ψ and $\nabla\Psi$, and are bounded by a polynomial in τ .

We shall now show explicitly that the evolution equations for $\delta\phi$ and δp take this form, and argue that the evolution equations for the other six functions $\delta\Lambda$, etc, do as well.

¹⁰In terms of $t = e^{-\tau}$, equation (28) takes the more familiar Fuchsian form

$$t\partial_t \Psi + A\Psi = -t^\mu F(t, x, \Psi, \nabla\Psi).$$

We start with the evolution equation for $\delta\phi$. Based on the Einstein Hamiltonian equation

$$\partial_\tau \phi = \frac{1}{4}p, \quad (31)$$

which we derive from the Hamiltonian (6) and (7), we obtain

$$\partial_\tau \delta\phi = \frac{1}{4}e^{-\epsilon\tau} \delta p \quad , \quad (32)$$

where we recall that ϵ is the strictly positive parameter appearing in the expansion ansatz (20)-(27). Clearly this is of the right form, with $\nu = 0$, $\mu_1 = \epsilon$, and $f_1 = \delta p$.

The equation for δp is not so simple. Based on the Einstein Hamiltonian equation

$$\partial_\tau p = 4e^{-2\tau} \left(e^\Lambda e^{ab} \phi_{,a} \right)_{,b} \quad (33)$$

we derive

$$\partial_\tau \delta p - \epsilon \delta p = 4 \left[e^{\overset{\circ}{\Lambda} + \epsilon\tau - v_\Lambda \tau + \delta\Lambda} e^{ab} \left(\overset{\circ}{\phi}_{,a} - v_{\phi,a} \tau + \delta\phi_{,a} \right) \right]_{,b} \quad (34)$$

with e^{ab} the inverse of e_{ab} from (5); i.e.,

$$e^{ab} = \frac{1}{2} \begin{pmatrix} e^{2z} + e^{-2z}(1-x)^2 & -e^{2z} - e^{-2z}(x^2-1) \\ -e^{2z} - e^{-2z}(x^2-1) & e^{2z} + e^{-2z}(1+x)^2 \end{pmatrix} \quad , \quad (35)$$

and with x and z to be expanded as in equations (20)-(27) and (12)-(19). Let us now write e^{ab} in the form

$$\begin{aligned} e^{ab} &= e^{-2z} \frac{1}{2} \begin{pmatrix} (1-x)^2 & -(x^2-1) \\ -(x^2-1) & (1+x)^2 \end{pmatrix} + e^{2z} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ &=: e^{-2z} e_I^{ab} + e^{2z} e_{II}^{ab} \end{aligned} \quad (36)$$

The right hand side of (34) splits, and we have

$$\begin{aligned} \partial_\tau p - \epsilon \delta p &= 4 \left\{ e^{\overset{\circ}{\Lambda} + \epsilon\tau - v_\Lambda \tau + \delta\Lambda} e^{-2\overset{\circ}{z} + 2v_z \tau - 2\delta z} e_I^{ab} \left[\overset{\circ}{\phi}_{,a} - v_{\phi,a} \tau + \delta\phi_{,a} \right] \right\}_{,b} \\ &\quad + 4 \left\{ e^{\overset{\circ}{\Lambda} + \epsilon\tau - v_\Lambda \tau + \delta\Lambda} e^{+2\overset{\circ}{z} - 2v_z \tau + 2\delta z} e_{II}^{ab} \left[\overset{\circ}{\phi}_{,a} - v_{\phi,a} \tau + \delta\phi_{,a} \right] \right\}_{,b} \end{aligned} \quad (37)$$

Calculating out the first term on the right hand side of equation (37), we obtain

$$\begin{aligned}
RHS_I = e^{(\epsilon - v_\Lambda + 2v_z)\tau} 4e^{\overset{\circ}{\Lambda} - 2\overset{\circ}{z}} e^{\delta\Lambda - 2\delta z} & \left\{ e_I^{ab},{}_b \left(\overset{\circ}{\phi}_{,a} - v_{\phi,a}\tau + \delta\phi_{,a} \right) \right. \\
& + e_I^{ab} \left(\overset{\circ}{\phi}_{,ab} - v_{\phi,ab}\tau + \delta\phi_{,ab} \right) \\
& + \left[\overset{\circ}{\Lambda}_{,b} - 2\overset{\circ}{z}_{,b} - (v_{\Lambda,b} - 2v_{z,b})\tau + \delta\Lambda_{,b} - 2\delta z_{,b} \right] \\
& \left. e_I^{ab} \left(\overset{\circ}{\phi}_{,a} - v_{\phi,a}\tau + \delta\phi_{,a} \right) \right\}. \quad (38)
\end{aligned}$$

This expression contains second derivatives of some of the fields included in Ψ , and so it cannot satisfy the conditions demanded for f_k from equation (30) as listed above. However, by introducing a new variable

$$\phi_a \equiv \phi_{,a} \quad (39)$$

together with its corresponding expansion with remainder term,

$$\phi_a = \overset{\circ}{\phi}_{,a} - v_{\phi,a}\tau + \delta\phi_a, \quad (40)$$

and by including $\delta\phi_a$ in an expanded version of Ψ , one readily transforms (37) into a function involving only first derivatives of the expanded Ψ , as required.

Now, inspecting the terms in (38), including those in the matrix e_I^{ab} , we see that, so long as the functions v_Λ and v_z satisfy the condition

$$v_\Lambda > 2v_z + \epsilon \quad (41)$$

for $\epsilon > 0$ as introduced above (equations (20)-(27)), then RHS_I takes the form required for the right hand side of (30), with $\mu_I := v_\Lambda - 2v_z - \epsilon$. Calculating the second term on the right hand side of (37), we get a very similar expression, only now we have $\mu_{II} := v_\Lambda + 2v_z - \epsilon$ and the condition for μ_{II} to be strictly positive is

$$v_\Lambda > -2v_z + \epsilon \quad . \quad (42)$$

Assuming that v_z is strictly positive (which is a requirement for this expansion ansatz), (41) is the more rigorous condition. Thus we find that in this

first step of checking whether solutions of the ansatz form (20)-(27) are to be AVTD, it is sufficient that condition (41) hold.

We proceed to consider the rest of the evolution equations for $\delta\Lambda$, δp_Λ , etc (including the added first derivative variables). We get very similar results. For example, the $\delta\Lambda$ evolution equation is

$$\partial_\tau \delta\Lambda = -e^{-\epsilon\tau} \delta p_\Lambda, \quad (43)$$

while the δp_Λ equation - derived from the evolution equation for p_Λ ,

$$\begin{aligned} \partial_\tau p_\Lambda = & -e^{-2\tau} \left\{ e^\Lambda e^{ab} \Lambda_{,ab} + (e^\Lambda e^{ab})_{,ab} \right. \\ & \left. + e^\Lambda [(e^{-2z})_{,u} x_{,v} - (e^{-2z})_{,v} x_{,u}] + 2e^\Lambda e^{ab} \phi_{,a} \phi_{,b} \right\} \end{aligned} \quad (44)$$

-can be written as

$$\begin{aligned} \partial_\tau \delta p_\Lambda - \epsilon \delta p_\Lambda = & -e^{(\epsilon - v_\Lambda + 2v_z)\tau} e^{\overset{\circ}{\Lambda} - 2\overset{\circ}{z}} e^{\delta\Lambda - 2\delta z} \left\{ e_I^{ab} \left[\overset{\circ}{\Lambda}_{,ab} - v_{\Lambda,ab} \tau + \delta\Lambda_{,ab} \right] \right. \\ & + e_I^{ab} \left(\overset{\circ}{\Lambda}_{,a} - v_{\Lambda,a} \tau + \delta\Lambda_{,a} \right) \left(\overset{\circ}{\Lambda}_{,b} - v_{\Lambda,b} \tau + \delta\Lambda_{,b} \right) \\ & + 2e_I^{ab} \left(\overset{\circ}{\phi}_{,a} - v_{\phi,a} \tau + \delta\phi_{,a} \right) \left(\overset{\circ}{\phi}_{,b} - v_{\phi,b} \tau + \delta\phi_{,b} \right) \\ & \left. + \text{similar terms} \right\} \\ & - e^{(\epsilon - v_\Lambda - 2v_z)\tau} e^{\overset{\circ}{\Lambda} + 2\overset{\circ}{z}} e^{\delta\Lambda + 2\delta z} \left\{ e_{II}^{ab} [\text{etc}] + \dots \right\}. \end{aligned} \quad (45)$$

We thus find that for this pair, too, so long as the inequality (41) holds, and so long as we define new field variables

$$\Lambda_a \equiv \Lambda_{,a}, \quad (46)$$

$$x_a \equiv x_{,a}, \quad (47)$$

$$z_a \equiv z_{,a}, \quad (48)$$

with corresponding expansions

$$\Lambda_a = \overset{\circ}{\Lambda}_{,a} - v_{\Lambda,a} \tau + \delta\Lambda_a, \quad (49)$$

$$x_a = \overset{\circ}{x}_{,a} + \delta x_a, \quad (50)$$

$$z_a = \overset{o}{z}_{,a} - v_{z,a} \tau + \delta z_a, \quad (51)$$

the evolution equations are of the proper (Fuchsian) form. The same inequality condition (41), together with the condition

$$v_z > \epsilon/4 \quad (52)$$

leads to the proper form for the evolution equations for the quantities δz , δp_z , δx , and δp_x , as well as for $\delta \phi_a$, $\delta \Lambda_a$, δx_a , and δz_a . The evolution equations for these latter four quantities are derived from their definitions. For example, the evolution equation for $\delta \phi_a$ is

$$\partial_\tau \delta \phi_a = \frac{1}{4} e^{-\epsilon \tau} \partial_a \delta p. \quad (53)$$

We have now verified the following:

Proposition 1 *If the polarized $U(1)$ Symmetric gravitational variables*

$$\mathcal{G}_{pol} := \{\phi, \phi_a, p, \Lambda, \Lambda_a, p_\Lambda, x, x_a, p_x, z, z_a, p_z\}$$

are expanded as in equations (12)-(19), (20)-(27), (40), and (49)-(51), then for any choice of the asymptotic data

$$\mathcal{A}_{pol} := \left\{ \overset{o}{\phi}, \overset{o}{\Lambda}, \overset{o}{x}, \overset{o}{z}, v_\phi, v_\Lambda, v_x, v_z \right\}$$

which satisfies the conditions

$$v_z > \epsilon/4 \quad (54)$$

and

$$v_\Lambda > 2v_z + \epsilon, \quad (55)$$

(for $\epsilon > 0$), together with the expansion ansatz condition

$$v_\phi > 0, \quad (56)$$

the vacuum Einstein evolution equations take the form of a Fuchsian system for

$$\delta \mathcal{G}_{pol} := \{\delta \phi, \delta \phi_a, \delta p, \delta \Lambda, \delta \Lambda_a, \delta p_\Lambda, \delta x, \delta x_a, \delta p_x, \delta z, \delta z_a, \delta p_z\} \quad .$$

It then follows from the work of Kichenassamy and Rendall [8] that we have

Corollary 1 *For each analytic choice of the asymptotic data \mathcal{A}_{pol} which satisfies conditions (54)-(56), there is a unique analytic solution $\delta\mathcal{G}_{pol}$ to the vacuum Einstein evolution equations for sufficiently large τ , with $\delta\mathcal{G}_{pol}$ approaching zero as τ approaches infinity.*

Hence we have an \mathcal{A}_{pol} -parameterized family of asymptotically velocity term dominated solutions to the $U(1)$ symmetric vacuum Einstein evolution equations.

We have so far ignored the Einstein constraint equations. We shall consider them in Section 4. In the rest of this section, we show that there is another asymptotic expansion ansatz which allows for nonzero r and w and yet still shows AVTD behavior. It includes the polarized ansatz as a subcase; and it introduces one extra free function into the asymptotic data, rather than two. Hence we call these “half-polarized” solutions.

The half-polarized solutions are defined by their asymptotic form, which appends the following expansions to those of the polarized solutions given in (12)-(19) and (20)-(27):

Expansion Ansatz (Half-Polarized)

$$r(u, v, \tau) = \overset{\circ}{r}(u, v) + e^{-\epsilon\tau} \delta r \quad (57)$$

$$w(u, v, \tau) = e^{4\left(\overset{\circ}{\phi} - v_{\phi}\tau + \delta\phi\right)} \left[-\frac{\overset{\circ}{r}}{4v_{\phi}} + \delta w \right] \quad . \quad (58)$$

These expansions introduce one new function, $\overset{\circ}{r}(u, v)$, into the set $\mathcal{A}_{1/2}$ of asymptotic data, and include two time-dependent expansion functions $\delta w(u, v, \tau)$ and $\delta r(u, v, \tau)$.

To verify that Proposition 1 and its Corollary 1 extend to the half-polarized ansatz, we first expand out the evolution equations for δw and δr . For δw , it follows from the Einstein Hamiltonian evolution equation

$$\partial_{\tau} w = e^{4\phi} r \quad (59)$$

that we have

$$\partial_\tau \delta w - 4v_\phi \delta w = e^{-\epsilon\tau} \left[\left(\frac{\overset{\circ}{r}}{4v_\phi} - \delta w \right) \delta p + \delta r \right] , \quad (60)$$

which is appropriate Fuchsian form, so long as $v_\phi \geq 0$ (a necessary condition for the expansion ansatz). For δr , it follows from the Einstein Hamiltonian evolution equation

$$\partial_\tau r = e^{-2\tau} \left(e^\Lambda e^{-4\phi} e^{ab} w_{,a} \right)_{,b} \quad (61)$$

that so long as condition (55) is satisfied, and so long as we introduce first derivative field variables for w , we have Fuchsian form as well.

We still need to verify that the evolution equations for $\delta\phi$, δp , and the other remaining functions in $\delta\mathcal{G}_{1/2}$ retain Fuchsian form. The evolution equations for $\delta\phi$, $\delta\Lambda$, δx , and δz , as well as for their first derivative field variables $\delta\phi_a$, $\delta\lambda_a$, δx_a and δz_a are unchanged by the nonvanishing of the r and w terms in the Hamiltonian (6)-(7), so they are in proper form. The evolution equations for δp , δp_Λ , δp_x , and δp_z all pick up extra terms which need to be checked. For example, the equation for δp , derived from the Einstein Hamiltonian evolution equation

$$\begin{aligned} \partial_\tau p &= 4e^{-2\tau} \left(e^\Lambda e^{ab} \phi_{,a} \right)_{,b} \\ &- 2e^{4\phi} r^2 + 2e^{-2\tau} e^\Lambda e^{-4\phi} e^{ab} w_{,a} w_{,b} \end{aligned} \quad (62)$$

takes the form

$$\begin{aligned} \partial_\tau \delta p - \epsilon \delta p &= 4 \left[e^{\overset{\circ}{\Lambda} + (\epsilon - v_\Lambda)\tau + \delta\Lambda} e^{ab} \left(\overset{\circ}{\phi}_{,a} - v_{\phi,a} \tau + \delta\phi_{,a} \right) \right]_{,b} \\ &- 2e^{4\overset{\circ}{\phi} + (\epsilon - 4\overset{\circ}{v}_\phi)\tau + 4\delta\phi} \left(\overset{\circ}{r} + e^{-\epsilon\tau} \delta r \right)^2 \\ &+ 2e^{\overset{\circ}{\Lambda} + (\epsilon - v_\Lambda)\tau + \delta\Lambda} e^{-4\overset{\circ}{\phi} + 4v_\phi\tau - 4\delta\phi} e^{ab} \times \\ &\times \left(e^{4\overset{\circ}{\phi} - 4v_\phi\tau + 4\delta\phi} \left[\frac{-\overset{\circ}{r}}{4v_\phi} + \delta w \right] \right)_{,a} \\ &\times \left(e^{4\overset{\circ}{\phi} - 4v_\phi\tau + 4\delta\phi} \left[\frac{-\overset{\circ}{r}}{4v_\phi} + \delta w \right] \right)_{,b} . \end{aligned} \quad (63)$$

The first term on the right hand side of equation (63) is just the right hand side of equation (34), the evolution equation for δp in the polarized

case; it has already been checked. The second term, a new one, clearly is in Fuchsian form so long as

$$v_\phi > \frac{1}{4}\epsilon \quad . \quad (64)$$

The third term can be written in the form

$$\begin{aligned} & 2e^{(\epsilon - v_\Lambda + 2v_z - 4v_\phi)\tau} e^{\overset{\circ}{\Lambda} - 2\overset{\circ}{z} + 4\overset{\circ}{\phi}} e^{\delta\Lambda - 2\delta z + 4\delta\phi} \times \\ & \times \left\{ e_I^{ab} \left[4 \left(\overset{\circ}{\phi}_{,a} - v_{\phi,a}\tau + \delta\phi_{,a} \right) \left(\frac{-\overset{\circ}{r}}{4v_\phi} + \delta w \right) + \left(\frac{-\overset{\circ}{r}_{,a}}{4v_\phi} + \frac{\overset{\circ}{r}}{4v_\phi^2} v_{\phi,a} + \delta w_{,a} \right) \right] \times \right. \\ & \quad \times \left[4 \left(\overset{\circ}{\phi}_{,b} - v_{\phi,b}\tau + \delta\phi_{,b} \right) \left(\frac{-\overset{\circ}{r}}{4v_\phi} + \delta w \right) + \left(\frac{-\overset{\circ}{r}_{,b}}{4v_\phi} + \frac{\overset{\circ}{r}}{4v_\phi^2} v_{\phi,b} + \delta w_{,b} \right) \right] \Big\} \\ & \quad + e^{(\epsilon - v_\Lambda - 2v_z - 4v_\phi)\tau} e^{\overset{\circ}{\Lambda} + 2\overset{\circ}{z} + 4\overset{\circ}{\phi}} e^{\delta\Lambda + 2\delta z + 4\delta\phi} \times \\ & \quad \times \{ e_{II}^{ab} [\text{same terms}]_a [\text{same terms}]_b \} \quad (65) \end{aligned}$$

So we find that this term—and consequently the half-polarized evolution equation for δp —takes Fuchsian form so long as

$$v_\Lambda > 2v_z - 4v_\phi + \epsilon \quad , \quad (66)$$

which follows from conditions (54)-(56). Examining the extra terms in the evolution equations for δp_Λ , δp_x , and δp_z , we reach the same conclusion. The full system of evolution equations (including the first derivative variables introduced above) takes Fuchsian form, and we have the following:

Proposition 2 *If the $U(1)$ Symmetric gravitational variables*

$$\mathcal{G}_{1/2} := \{\phi, \phi_a, p, \Lambda, \Lambda_a, p_\Lambda, x, x_a, p_x, z, z_a, p_z, w, w_a, r\}$$

are expanded as in equations (12)-(27), then for any choice of the asymptotic data

$$\mathcal{A}_{1/2} := \left\{ \overset{\circ}{\phi}, \overset{\circ}{\Lambda}, \overset{\circ}{x}, \overset{\circ}{z}, v_\phi, v_\Lambda, v_x, v_z, \overset{\circ}{r} \right\}$$

which satisfies conditions (54)-(56), the vacuum Einstein evolution equations take the form of a Fuchsian system for

$$\delta\mathcal{G}_{1/2} := \{\delta\phi, \delta\phi_a, \delta p, \delta\Lambda, \delta\Lambda_a, \delta p_\Lambda, \delta x, \delta x_a, \delta p_x, \delta z, \delta z_a, \delta p_z, \delta w, \delta w_a, \delta r\}$$

Corollary 2 *For each analytic choice of the asymptotic data $\mathcal{A}_{1/2}$ which satisfies (54)-(56), there is a unique solution $\delta\mathcal{G}_{1/2}$ to the vacuum Einstein evolution equations for large τ , with $\delta\mathcal{G}_{1/2}$ approaching zero as τ approaches infinity.*

4 The Constraint Equations

The results of section 3 show that for the polarized as well as for the half-polarized families of $U(1)$ Symmetric spacetimes, there is a function space of asymptotic data such that for any choice of data in that function space there is a unique corresponding solution of the $U(1)$ Symmetric Einstein evolution equations which is AVTD and asymptotically approaches that choice of data. These results ignore the Einstein constraint equations; hence the spacetimes they describe are generally not solutions of the Einstein vacuum field equations. We now wish to show that for both of these families, there is a set of constraint equations *on the asymptotic data* such that if a set of asymptotic data satisfies these asymptotic constraint equations, then the corresponding solution of the evolution equations satisfies the full constraint equations (and therefore satisfies the full Einstein vacuum field equations). In addition, we wish to show that, for each of the two families, the asymptotic constraint equations admit a set of solutions of the appropriate number of parameters. We may then conclude that we have a corresponding set of AVTD solutions of the Einstein vacuum field equations in each of the two cases.

We note that, in addition to the constraint equations $\mathcal{H}_0 = 0$, $\mathcal{H}_u = 0$, and $\mathcal{H}_v = 0$, which we treat here, there are new constraints on the initial data of the form $\phi_a = \partial_a \phi$ which are necessitated by the introduction of the new variables ϕ_a , λ_a , x_a , and z_a , as discussed above. These readily translate into constraints on the asymptotic data, and are preserved by the evolution equations, so we shall not treat them here any further.

We begin the analysis of the constraints $\mathcal{H}_0 = 0$, $\mathcal{H}_u = 0$, and $\mathcal{H}_v = 0$ by deriving the corresponding asymptotic constraint equations. We do this for each family by substituting the appropriate expansion ansatz for the fields into the constraint equations, and then letting $\tau \rightarrow \infty$, with the consequent vanishing of all $\delta(\text{field})$ terms and all terms consisting of decaying exponentials times fields bounded by polynomials in τ . More specifically, starting with the polarized spacetimes, we substitute expressions (20)-(27) into (8), (9), and (10). Then, setting δx , δz , $\delta \phi$, $\delta \Lambda$ and their derivatives

to zero, and setting $e^{-4v_z\tau}$ and like terms to zero as well, we obtain the following¹¹.

Asymptotic Constraint Equations

$$\begin{aligned} 0 &= \overset{\circ}{\mathcal{H}}_0 \\ &:= 2 \left[(v_z)^2 + (v_\phi)^2 - \frac{1}{4} (v_\Lambda)^2 \right] \end{aligned} \quad (67)$$

$$\begin{aligned} 0 &= \overset{\circ}{\mathcal{H}}_u \\ &:= -4v_z \overset{\circ}{z}_{,u} + v_x \overset{\circ}{x}_{,u} + v_\Lambda \overset{\circ}{\Lambda}_{,u} - 4v_\phi \overset{\circ}{\phi}_{,u} - v_{\Lambda,u} \\ &\quad + \frac{1}{2} \left\{ - \left(1 + \overset{\circ}{x} \right)^2 v_x + 4 \left(1 + \overset{\circ}{x} \right) v_z \right\}_{,v} \\ &\quad - \frac{1}{2} \left\{ + \left(1 - \overset{\circ}{x}^2 \right) v_x + 4 \overset{\circ}{x} v_z \right\}_{,u} \end{aligned} \quad (68)$$

$$\begin{aligned} 0 &= \overset{\circ}{\mathcal{H}}_v \\ &:= -4v_z \overset{\circ}{z}_{,v} + v_x \overset{\circ}{x}_{,v} + v_\Lambda \overset{\circ}{\Lambda}_{,v} - 4v_\phi \overset{\circ}{\phi}_{,v} - v_{\Lambda,v} \\ &\quad + \frac{1}{2} \left\{ \left(1 - \overset{\circ}{x} \right)^2 v_x + 4 \left(1 - \overset{\circ}{x} \right) v_z \right\}_{,u} \\ &\quad + \frac{1}{2} \left\{ \left(1 - \overset{\circ}{x}^2 \right) v_x + 4 \overset{\circ}{x} v_z \right\}_{,v} \end{aligned} \quad (69)$$

These are constraints on the choice of the asymptotic data \mathcal{A}_{pol} for polarized $U(1)$ Symmetric spacetimes. A similar procedure leads to constraints on $\mathcal{A}_{1/2}$, the family of half-polarized $U(1)$ Symmetric spacetimes. One finds that the asymptotic constraints here are exactly (67) -(69); the terms $\frac{1}{2}e^{4\phi}r^2$ and $\frac{1}{2}e^\Lambda e^{-4\phi}e^{ab}w_{,a}w_{,b}$ in \mathcal{H} and the terms $rw_{,u}$ in \mathcal{H}_u and $rw_{,v}$ in \mathcal{H}_v have no asymptotic effect if one expands r and w as in (57) and (58).

We now want to argue that if the asymptotic data \mathcal{A}_{pol} , or $\mathcal{A}_{1/2}$ satisfy the asymptotic constraints (67)-(69) and if the corresponding fields \mathcal{G}_{pol} or $\mathcal{G}_{1/2}$ satisfy the $U(1)$ Symmetric evolution equations generated by (6), then

¹¹In calculating $\overset{\circ}{\mathcal{H}}_u$ and $\overset{\circ}{\mathcal{H}}_v$, we have used (67) to cancel certain terms and thereby simplify the expressions (68) and (69)

\mathcal{G}_{pol} , and $\mathcal{G}_{1/2}$ satisfy the initial value constraints (8)-(10) on any Cauchy surface sufficiently close to the singularity (i.e., for sufficiently large τ). We argue this as follows (focusing first on the polarized case): If we substitute the expansion ansatz (20)-(27) for the polarized fields into the expressions (8)-(10) for the constraints, then we can write

$$\mathcal{H}_0 = \overset{\circ}{\mathcal{H}}_0 + e^{-\mu\tau} \delta\mathcal{H}_0 \quad (70)$$

$$\mathcal{H}_u = \overset{\circ}{\mathcal{H}}_u + \delta\mathcal{H}_u \quad (71)$$

$$\mathcal{H}_v = \overset{\circ}{\mathcal{H}}_v + \delta\mathcal{H}_v \quad (72)$$

where $\{\mathcal{H}_0, \mathcal{H}_u, \mathcal{H}_v\}$ are the asymptotic constraint expressions (67)-(69), μ is a strictly positive constant, and $\{\delta\mathcal{H}_0, \delta\mathcal{H}_u, \delta\mathcal{H}_v\}$ are the remainder terms (defined by (70)-(72)). Our calculations leading to (67)-(69) show that, regardless of whether the asymptotic data \mathcal{A}_{pol} satisfy the constraints (67)-(69), if the fields \mathcal{G}_{pol} evolve via the $U(1)$ Symmetric evolution equations, then the remainder terms approach zero as $\tau \rightarrow \infty$. We want to show more. We will show that in fact, so long as the asymptotic constraints (67)-(69) are satisfied, \mathcal{H}_0 , \mathcal{H}_u , and \mathcal{H}_v are identically zero for sufficiently large τ . We do this by using the $U(1)$ Symmetric evolution equations to show that $\{\mathcal{H}_0, \mathcal{H}_u, \mathcal{H}_v\}$ are identically zero for sufficiently large τ . To do that, we use the $U(1)$ Symmetric evolution equations to show that $\delta\mathcal{H}_0, \delta\mathcal{H}_u, \delta\mathcal{H}_v$ satisfy a Fuchsian system, and then note that this Fuchsian system admits $\delta\mathcal{H}_0 = 0$, $\delta\mathcal{H}_u = 0$, $\delta\mathcal{H}_v = 0$ as a solution. Thus, since the solution to the Fuchsian system (for a given set of asymptotic data) is unique near $\tau \rightarrow \infty$, and since $\{0, 0, 0\}$ is a solution, it follows that we have $\delta\mathcal{H}_0 = 0$, $\delta\mathcal{H}_u = 0$, $\delta\mathcal{H}_v = 0$ for sufficiently large τ . To complete the argument, we note that if the asymptotic data \mathcal{A}_{pol} satisfy the asymptotic constraints and if $\{\delta\mathcal{H}_0, \delta\mathcal{H}_u, \delta\mathcal{H}_v\} = \{0, 0, 0\}$ for large τ , then it follows from (70)-(72) that $\mathcal{H}_0 = 0$, $\mathcal{H}_u = 0$, $\mathcal{H}_v = 0$ for large τ . The constraints are satisfied.

The key to this argument is the derivation of the Fuchsian system for $\{\delta\mathcal{H}_0, \delta\mathcal{H}_u, \delta\mathcal{H}_v\}$. To obtain this system, we start with the evolution equations for $\{\mathcal{H}_0, \mathcal{H}_u, \mathcal{H}_v\}$, which can be calculated from the divergence Bianchi

identity:¹²

$$\partial_\tau \mathcal{H}_0 = -e^{\Lambda-2\tau} [\Lambda_{,b} e^{ab} (\mathcal{H}_a - \beta_a \mathcal{H}_3) + (e^{ab} (\mathcal{H}_a - \beta_a \mathcal{H}_3))_{,b}] \quad (73)$$

$$\partial_\tau \mathcal{H}_a = -\partial_a \mathcal{H}_0 \quad (74)$$

where a, b run over u and v . If we now substitute the expansions (70)-(72) into these evolution equations, and further set $\overset{\circ}{\mathcal{H}}_0, \overset{\circ}{\mathcal{H}}_u, \overset{\circ}{\mathcal{H}}_v$ and \mathcal{H}_3 equal to zero, then we derive

$$\partial_\tau \delta \mathcal{H}_0 - \mu \delta \mathcal{H}_0 = (e^{\Lambda-2\tau+\mu\tau}) [\Lambda_{,b} e^{ab} \delta \mathcal{H}_a + (e^{ab} \delta \mathcal{H}_a)_{,b}] \quad (75)$$

$$\partial_\tau \delta \mathcal{H}_a = -e^{-\mu\tau} \partial_a (\delta \mathcal{H}_0) \quad (76)$$

These equations clearly constitute a Fuchsian system for $\{\delta \mathcal{H}_0, \delta \mathcal{H}_u, \delta \mathcal{H}_v\}$ so long as $\mu > 0$ and $v_\Lambda > 2v_z + \mu$, which are the familiar inequality conditions on the asymptotic data. Further, we note that $\{\delta \mathcal{H}_0, \delta \mathcal{H}_u, \delta \mathcal{H}_v\} = \{0, 0, 0\}$ is a solution of the system (75)-(76). Finally, we note that (75)-(76) holds not just for the polarized $U(1)$ Symmetric spacetimes, but in fact for the half-polarized $U(1)$ Symmetric spacetimes as well. Thus we have proven the following:

Proposition 3 *If the asymptotic data for either a polarized or a half-polarized $U(1)$ Symmetric spacetime is analytic and satisfies the asymptotic constraint equations (67)-(69), then the corresponding spacetime satisfies the full set of Einstein constraint equations—and hence the full set of Einstein vacuum field equations—for sufficiently large τ .*

While Proposition 3 tells us that any solution of the asymptotic constraint equations (67)-(69) leads to a solution¹³ of the vacuum field equations, it does not tell us anything about finding and parameterizing solutions of the asymptotic constraints. We could work directly with (67)-(69); however,

¹²Note that while \mathcal{H}_0 is a weight two scalar density, $\mathcal{H}_u, \mathcal{H}_v$ and \mathcal{H}_3 are all weight one scalar densities. Note also that while we have already eliminated $\mathcal{H}_3 = 0$ (see equation (3)), we leave \mathcal{H}_3 in equations (74)-(76) for the sake of generality.

¹³In fact, since r_0 and w_0 do not appear in the asymptotic constraints, every solution of (67)-(69) leads to a large set of half-polarized solutions, along with one polarized solution.

since they are specified on a possibly degenerate manifold, we choose an alternative approach.

The alternative approach starts with the recognition that (67)-(69) are the asymptotic constraint equations for the $U(1)$ Symmetric VTD equations as well as for the $U(1)$ Symmetric vacuum Einstein equations. To see this, we substitute the large τ VTD solutions—(12)-(19) for the polarized case, and (12)-(19) together with $r = \overset{\circ}{r}$ and $w = 0$ for the half-polarized case—into the VTD constraint equations:

$$\begin{aligned} 0 &= \hat{\mathcal{H}}_0 \\ &= \frac{1}{8}p_z^2 + \frac{1}{2}e^{4z}p_x^2 + \frac{1}{8}p^2 + \frac{e^{4\phi}}{2}r^2 - \frac{1}{2}p_\Lambda^2 \end{aligned} \quad (77)$$

$$\begin{aligned} 0 &= \hat{\mathcal{H}}_u \\ &= \mathcal{H}_u \quad (\text{see equation (9)}) \end{aligned} \quad (78)$$

$$\begin{aligned} 0 &= \hat{\mathcal{H}}_v \\ &= \mathcal{H}_v \quad (\text{see equation (10)}) \end{aligned} \quad (79)$$

We obtain, in both cases, equations (67)-(69).

We next note that we have global existence in time τ for solutions of the $U(1)$ Symmetric VTD evolution equations. This property holds even for the slightly generalized form of these equations, which we obtain from the Hamiltonian

$$\hat{H} = \int_{T^2} \left[\alpha \hat{\mathcal{H}}_0 + 2p_\Lambda \right] dudv, \quad (80)$$

where α is a freely specifiable “lapse” function. Global existence is an immediate consequence of the following expression for the general solution to these evolution equations (clearly well behaved for all values of $\tau \in (-\infty, +\infty)$) [2] [6]:

$$\phi = -v_\phi (\alpha\tau - \tau_\phi) - \frac{1}{2}\ell n \left[|\zeta_\phi| \left(1 + e^{-4v_\phi(\alpha\tau - \tau_\phi)} \right) \right] \quad (81)$$

$$p = -4v_\phi \left(\frac{1 - e^{-4v_\phi(\alpha\tau - \tau_\phi)}}{1 + e^{-4v_\phi(\alpha\tau - \tau_\phi)}} \right) \quad (82)$$

$$w = \xi_\phi - \left[\zeta_\phi \left(1 + e^{-4v_\phi(\alpha\tau - \tau_\phi)} \right) \right]^{-1} \quad (83)$$

$$r = -4\zeta_\phi v_\phi \quad (84)$$

$$x = \xi_z - \left[\zeta_z \left(1 + e^{-4v_z(\alpha\tau - \tau_z)} \right) \right]^{-1} \quad (85)$$

$$p_x = -4\zeta_z v_z \quad (86)$$

$$z = -v_z(\alpha\tau - \tau_z) - \frac{1}{2}\ell n \left[|\zeta_z| \left(1 + e^{-4v_z(\alpha\tau - \tau_z)} \right) \right] \quad (87)$$

$$p_z = -4v_z \left(\frac{1 - e^{-4v_z(\alpha\tau - \tau_z)}}{1 + e^{-4v_z(\alpha\tau - \tau_z)}} \right) \quad (88)$$

$$\Lambda = \Lambda_0 + 2\tau - \alpha v_\Lambda \tau \quad (89)$$

$$p_\Lambda = v_\Lambda \quad (90)$$

Here $\tau_\phi, v_\phi > 0$, $\zeta_\phi, \xi_\phi, \tau_z, v_z > 0$, $\zeta_z, \xi_\phi, \Lambda_0$ and v_Λ are all free functions of u and v ; they are “constants of integration” for the set of *ordinary* differential equations which comprise the $U(1)$ Symmetric VTD evolution equation system. Note that (81)-(90) is the general solution for the *full* $U(1)$ symmetric VTD evolution system; no polarization or half-polarization assumption has been made. Note also that the positivity conditions on v_ϕ and v_z are necessary for this parametrization of the solutions of the VTD equations.

With global existence of the VTD solutions established, we may consider solutions of the VTD constraints (77)-(79) at finite times τ , at which there is no difficulty with degeneracy of the spatial geometry. To relate solutions of (77)-(79) to solutions of the asymptotic VTD constraints (67)-(69) (the same as the asymptotic Einstein constraints), it is useful to establish the following

Proposition 4 *The VTD constraint functions $\hat{\mathcal{H}}_0$, $\hat{\mathcal{H}}_u$, and $\hat{\mathcal{H}}_v$, when evaluated for solutions (81)-(90) of the VTD evolution equations, are independent of τ , so long as for some τ_0 , $\hat{\mathcal{H}}_0(\tau_0) = 0$*

Proof: The time independence of the quantity $\hat{\mathcal{H}}_0$ may be established directly, by substituting expressions (81)-(90) for the fields $\{\phi(\tau), p(\tau), w(\tau), r(\tau), x(\tau), p_x(\tau), z(\tau), p_z(\tau), \Lambda(\tau), p_\Lambda(\tau)\}$, into the expression (77) for $\hat{\mathcal{H}}_0$. One obtains, for all τ (and all α), $2v_z^2 + 2v_\phi^2 - \frac{1}{2}v_\Lambda^2$, which is independent of τ (and equal to $\hat{\mathcal{H}}_0$).

Rather than proceeding to substitute (81)-(90) into (78) and (79), we may establish the τ independence of $\hat{\mathcal{H}}_u$ and $\hat{\mathcal{H}}_v$ using the following argument: Since $\hat{\mathcal{H}}_u = \mathcal{H}_u$ and $\hat{\mathcal{H}}_v = \mathcal{H}_v$, we find that $\hat{\mathcal{H}}_u$ and $\hat{\mathcal{H}}_v$ both generate spatial diffeomorphisms, tangent to $\tau = \text{constant}$ surfaces. With $\hat{\mathcal{H}}_0$ constant and presumed zero at time τ_0 , we have $\hat{\mathcal{H}}_0 = 0$ for all time. Hence for any vector field $M^a(x)$ on T^2 , we have

$$\left\{ \int_{T^2} M^a(x) \hat{\mathcal{H}}_a(x) dx, \int_{T^2} \alpha(y) \hat{\mathcal{H}}_0(y) dy \right\} = 0 \quad (91)$$

where $\{, \}$ indicates the Poisson bracket.

The generalized VTD Hamiltonian \hat{H} , from equation (80), generates VTD time evolution for $\hat{\mathcal{H}}_u$ and $\hat{\mathcal{H}}_v$ (as well as for any other function of the VTD fields). As argued above, $\hat{\mathcal{H}}_u$ and $\hat{\mathcal{H}}_v$ commute with $\int \alpha \hat{\mathcal{H}}_0$. Calculating the Poisson bracket of $\hat{\mathcal{H}}_u$ and $\hat{\mathcal{H}}_v$ (times an arbitrary M^a , integrated over T^2), with the remaining piece of the Hamiltonian \hat{H} , we have

$$\begin{aligned} & \left\{ \int_{T^2} M^a(x) \hat{\mathcal{H}}_a(x) dx, \quad 2 \int_{T^2} p_\Lambda(y) dy \right\} \\ &= \left\{ - \int_{T^2} (M^a p_\Lambda)_{,a} \Lambda dx, 2 \int_{T^2} p_\Lambda(y) dy \right\} \\ &= -2 \int_{T^2} (M^a p_\Lambda)_{,a} dx \\ &= 0 \end{aligned} \quad (92)$$

Thus $\hat{\mathcal{H}}_u$ and $\hat{\mathcal{H}}_v$ commute with the VTD Hamiltonian, from which it follows that $\hat{\mathcal{H}}_u$ and $\hat{\mathcal{H}}_v$ are VTD constant.

Our remaining task now is to show that one can find a full set of solutions of the VTD constraint equations (77)-(79) – five¹⁴ free functions on T^2 in the polarized case; and six free functions as T^2 in the half-polarized case. To do this, it is useful to first show the following: For any fixed choice of the

¹⁴Three of these functions are related to coordinate choice, and thus are not physical.

asymptotic data \mathcal{A}_{pol} (or $\mathcal{A}_{1/2}$) there is a choice of the lapse function α for which the VTD fields at $\tau = 1$ have constant mean curvature (CMC) in a $2 + 1$ dimensional sense; i.e., based on the $2 + 1$ Lorentz signature metric

$$\gamma = -\alpha^2 e^{2\Lambda-4\tau} d\tau^2 + e^{-2\tau+\Lambda} e_{ab} dx^a dx^b. \quad (93)$$

(compare with the 3+1 metric (1).) The spatial volume element for γ from (93) is

$${}^2\mu = e^{\Lambda-2\tau} \quad (94)$$

Using the VTD evolution equations and solutions to calculate the mean extrinsic curvature for a $\tau = \text{constant}$ surface in this $2 + 1$ dimensional space-time, we obtain

$$\begin{aligned} trK &= \frac{-1}{(\text{lapse})} \frac{\partial_\tau ({}^2\mu)}{{}^2\mu} \\ &= \frac{v_\Lambda}{e^{\Lambda-2\tau}} \\ &= \frac{v_\Lambda}{e^{\Lambda_0-\alpha\tau v_\Lambda}}. \end{aligned} \quad (95)$$

If we now set $\tau = 1$ and $trK = C > 0$ in (95), we can solve for α :

$$\alpha = \frac{1}{v_\Lambda} (\Lambda_0(x) - \ell n(v_\Lambda(x)/C)) \quad (96)$$

Noting that our main results concerning the evolution behavior of the fields (Corollaries 1,2, and 3) hold only if v_Λ is positive definite, we see that indeed, for any fixed data \mathcal{A}_{pol} or \mathcal{A}_r , we can choose α (as in (96)) so that the $\tau = 1$ hypersurface has constant mean curvature. We ensure the positivity of α by a sufficiently large choice of the constant C .

Now that we know that any VTD solution admits a hypersurface with constant mean curvature in the 2+1 sense described above, we can use this CMC condition to help in the analysis of the VTD constraint equations, and show that we have a full set of solutions. We carry out this analysis using the conformal method, adapted to $U(1)$ Symmetric data in 2+1 form, as in [20] (where it is applied to the Einstein constraints). Specifically, on T^2 , the manifold transverse to the $U(1)$ orbits, we choose for our conformal data (1) a Riemannian metric λ_{ab} , (2) a symmetric tensor σ^{cd} which is divergence-free

and trace-free with respect to λ_{ab} , (3) a constant τ representing the mean extrinsic curvature of the geometry on T^2 , and (4) a pair of functions $\tilde{\phi}$ and \tilde{p}_ϕ which are treated as matter fields, but actually correspond to the geometry in the direction tangent to the orbits of the $U(1)$ isometry group. We then attempt to solve a system of three equations for a vector field W (which generates the longitudinal part of the extrinsic curvature on T^2) and a positive definite conformal factor ψ . If indeed for the given set of conformal data $\{\lambda, \sigma, \tau, \tilde{\phi}, \tilde{p}_\phi\}$, solutions W and ψ exist, then we may construct from W and ψ and the conformal data a set of initial data $\{\gamma, \pi, \phi, p_\phi\}$, which satisfies the constraints.

The equations to be solved for W and ψ are generally coupled, and it is generally not easy to determine if solutions exist. For CMC conformal data, however, the equations decouple; and in the case of the Einstein constraints, it has been determined that solutions to the equations exist for essentially all choices of the conformal data. [20]

For the VTD constraints, of interest here, this decoupling still holds. While there has not been any systematic study of the equations for W and ψ resulting from the VTD constraints, such a study is easily carried out. Indeed, the supermomentum constraints are the same for the Einstein and the VTD cases; in both cases, we have a linear elliptic system to be solved for W , and in both cases a solution exists for all choices of conformal data satisfying the integrability condition

$$\int \left(\tilde{p} Y^a \nabla_a \tilde{\phi} \right) = 0 \quad (97)$$

where Y is any conformal Killing field on T^2 . The superHamiltonian constraint, to be solved for ψ , is different for the Einstein and the VTD cases. In the former case, it is a nonlinear elliptic equation for ψ , while in the latter case, it is an algebraic equation for ψ . In both cases, one verifies that for nonzero constant mean curvature and for generic choices of the conformal data, a unique solution exists. We note that if the conformal data is chosen to be analytic, it follows from the ellipticity of the constraint equations that the corresponding initial data is analytic. It further follows from the VTD evolution equations and their general solution that the corresponding asymptotic data is analytic.

The analysis which we have just sketched shows that we have a full set of analytic solutions of the VTD constraint equations, and hence have a full set of analytic solutions of the asymptotic Einstein constraints. If such

solutions are to correspond to AVTD solutions of the Einstein equations, we must verify that the asymptotic data satisfies the inequalities $v_z > \epsilon/4$, $v_\phi > \epsilon/4$, and $v_\Lambda > 2v_z + \epsilon$, which are sufficient for the evolution equations to be Fuchsian.

General solutions of the asymptotic constraints do *not* satisfy these inequalities. We claim, however, that there are open sets of choices of the conformal data which do guarantee that these inequalities hold. To argue this, we first note that as a consequence of the asymptotic constraint equation (67) $v_\Lambda^2 = 4(v_z^2 + v_\phi^2)$, of the compactness of T^2 , and of our freedom to choose ϵ to be any positive constant, all of the necessary inequalities (54)-(56) hold (for some $\epsilon > 0$) so long as v_ϕ and v_z are both positive definite.

Now, in finding data which satisfy the VTD constraints at finite time ($\tau = 1$), we have no apparent direct control over the values of the data for arbitrarily large τ (ie, the values of the asymptotic data). However, examining the explicit form of the global-in-time VTD solutions (81-90), we find that with $\tau = 1$ and $\tau_\phi = 0 = \tau_z$, we have

$$p = -4v_\phi \tanh(2\alpha v_\phi) \quad (98)$$

and

$$p_z = -4v_z \tanh(2\alpha v_z) \quad (99)$$

It follows from (98)-(99) that for any set of data $\{\phi, \Lambda, r, x, z, p, p_\Lambda, \omega, p_x, p_z\}$ at $\tau = 1$ which has $p < 0$ and $p_z < 0$ for all u, v in T^2 , there exists a unique choice of positive definite functions v_ϕ and v_z such that (54)-(56) hold. (The same is true for nonzero, but sufficiently small, values of τ_ϕ and τ_z .) Thus each solution of the VTD constraints at $\tau = 1$ which has $p < 0$ and $p_z < 0$ has corresponding asymptotic data which satisfies the inequalities (54)-(56).

The conformal procedure for solving the VTD constraints, sketched above, allows us to freely choose the function p at $\tau = 1$, and hence its sign; however, since the quantity p_z is part of the extrinsic curvature of the two dimensional geometry in our $2+1$ treatment of the constraints, we cannot directly choose p_z . Yet we can argue as follows that we can control the sign of p_z : As just noted, p_z is related to the extrinsic curvature of the geometry on T^2 . More explicitly, letting π^{ab} denote the momentum representation of this extrinsic curvature, we have

$$p_z = [\pi^{uu}(e^{2z} - e^{-2z}(1+x)^2) + \pi^{vv}(e^{2z} - e^{-2z}(1-x)^2) + 2\pi^{uv}(e^{2z} - e^{-2z}(x^2 - 1))]e^{\Lambda-2\tau} \quad (100)$$

Now in solving the momentum constraints via the conformal method, one determines the vector field W , which partially determines π^{ab} . However, one can always add to this part of π^{ab} an arbitrary divergence-free, trace-free, symmetric tensor density σ^{ab} . Since all metrics on T^2 are conformally related to a flat metric, one finds that in appropriate coordinates, σ^{ab} is divergence-free and trace-free if and only if σ_b^a is a spatially constant trace-free tensor density. It follows from this fact, and from equation (99), that through the choice of σ^{ab} , one can always guarantee that p_z is negative definite. We then obtain, as argued above, the positive definiteness of v_z .

We conclude from this argument that, while it is not true that all sets of VTD data satisfying the VTD constraints at time $\tau = 1$ lead to asymptotic data satisfying the inequalities, if we impose certain open conditions on the choice of the conformal data, then we do obtain VTD data whose asymptotic data satisfies the inequalities. Noting the invertibility of functions such as $f(v) = -4 \tanh(2\alpha v)$ for $f < 0$ and $\alpha v > 0$, we readily verify that for both the polarized and the half-polarized $U(1)$ Symmetric spacetimes, we have a full set of asymptotic data—two free functions for the polarized solutions and three free functions for the half polarized solutions (after quotienting out the diffeomorphism gauge freedom on T^2)—which satisfy the asymptotic constraints as well as the inequalities which we have found to be sufficient to guarantee that the evolution equations are Fuchsian. Combining this result with those of Section 3 we have

Theorem 1 *There is a family of $U(1)$ Symmetric solutions of the vacuum Einstein equations on $T^3 \times R$ which is AVTD with respect to a harmonic time foliation, is characterized by analytic asymptotic data $\mathcal{A}_{1/2}$ satisfying the asymptotic constraints (67)-(69), and is parametrized by three free functions on T^2 .*

Those members of this family of spacetimes which have asymptotic data with $\overset{\circ}{r} = 0$ are AVTD solutions of the polarized $U(1)$ Symmetric vacuum Einstein equations. This polarized subfamily is parametrized by two free functions on T^2 .

5 AVTD Behavior and the Choice of Observers

The definition of AVTD behavior which we use here depends on a choice of observers; or equivalently, on a choice of coordinates. One may therefore ask if alternative sets of observers will agree on the presence or absence of AVTD behavior in a given spacetime. This issue has not yet been carefully addressed, since in most previous studies [8] [9] [11], the family of spacetimes under consideration and the analytic techniques used have singled out a particular choice of time foliation and time threading. For the present study of $U(1)$ Symmetric spacetimes, however, our choice of time is a bit more flexible, so we may begin to address the issue of the dependence of the verification of AVTD behavior on the choice of spacetime observers.

As noted earlier, we use “harmonic time” in working with the $U(1)$ Symmetric spacetimes, which means that the time function τ satisfies the wave equation $\square\tau = 0$. This condition does not fix the choice of time foliation; we may freely choose an initial Cauchy surface, along with an initial choice of lapse function (This amounts to Cauchy data for the function τ). The condition $\square\tau = 0$ then determines the rest of the time foliation.

Fixing the time foliation does not necessarily fix the timelike observers. They are determined by the choice of the spacetime “threading”; ie, the choice of a congruence of timelike paths (corresponding to the worldlines of the observers). Our analysis is considerably simplified, however, if we require that the observer paths be everywhere orthogonal to the leaves of the spacetime foliation (so the shift vector field is everywhere zero). With this condition imposed, the choice of spacetime observers is fixed by the choice of the (harmonic) time function τ .

While it would be useful to compare the observations relevant to AVTD behavior that are seen by each set of observers corresponding to each choice of a harmonic time in a given $U(1)$ Symmetric spacetime, we shall here pursue a more modest goal: Given a fixed polarized $U(1)$ Symmetric spacetime and a fixed choice of harmonic time such that the corresponding surface-orthogonal observers see AVTD behavior, we shall show that there is a full (two free functions on T^2) family of other harmonic time choices ¹⁵ such that their corresponding surface-orthogonal observers see AVTD behavior as well.

¹⁵We call the parametrization of this family of time choices by two free functions “full” since the space of Cauchy data for solutions of $\square\tau = 0$ consists of two functions.

Note that while we focus on the polarized case here to simplify the discussion, the same sort of results hold for half-polarized $U(1)$ Symmetric solutions as well.

We start by identifying the family of alternate harmonic time choices. To do this, we fix a polarized $U(1)$ Symmetric spacetime $(T^3 \times R, g)$ and a harmonic time choice τ whose corresponding observers see AVTD behavior. We then write the harmonic time condition for a new time function T ; in first order form, with $\zeta := \partial_\tau T$, we have

$$\partial_\tau T = \zeta \quad (101)$$

$$\partial_\tau \zeta = (e^{-2\tau} e^\Lambda e^{ab} T_{,a})_{,b}. \quad (102)$$

(Here we are choosing to work with the $2+1$ rather than $3+1$ version of the harmonic time coordinate). Now noting that

$$\hat{T} = a(x) + b(x)\tau \quad (103)$$

$$\hat{\zeta} = b(x) \quad (104)$$

for arbitrary functions $a(x)$ and $b(x) > 0$ on T^2 solves the VTD version of equations (101)-(102) (which sets the right hand side of equation (102) to zero), we seek a family of solutions of (101)-(102) of the form

$$T = a(x) + b(x)\tau + \delta T \quad (105)$$

$$\zeta = b(x) + e^{-\epsilon\tau} \delta\zeta \quad (106)$$

for some $\epsilon > 0$. (The condition $b > 0$ is required so that T uniformly approaches infinity as τ approaches infinity). Plugging these forms into (101)-(102), we derive

$$\partial_\tau \delta T = e^{-\epsilon\tau} \delta\zeta \quad (107)$$

and

$$\partial_\tau \delta\zeta - \epsilon\delta\zeta = e^{(\epsilon - v_\Lambda + 2v_z)\tau} f(\tau, u, v, T, T_\nabla), \quad (108)$$

where T_∇ denotes a set of new variables which we introduce so that only first derivatives appear, and where the function f is analytic in $(u, v, \tau, T, T_\nabla)$ and bounded by a polynomial in τ . Thus, presuming the usual restrictions on v_Λ and v_z , we have a Fuchsian system for δT and $\delta\zeta$ such that T and ζ of the form (105) and (106) solve (101)-(102). This gives us our family (parametrized by the two functions $a(u, v)$ and $b(u, v) > 0$) of choices of harmonic time for our fixed spacetime $(T^3 \times R, g)$.

Before addressing the question of observer-dependence of AVTD behavior for these different foliations, we wish to compare the paths of the observers orthogonal to the $T = a + b\tau + \delta T$ foliation with those orthogonal to the τ foliation. One way to do this is to start with the metric $g(x^i, \tau)$ expressed in terms of the τ foliation and the observers x^i orthogonal to the τ foliation, then re-express g in terms of the T foliation but with the τ -compatible observers retained, and finally determine the shift vector $M(x, T)$, which measures the extent to which the τ -compatible observers fail to be orthogonal to the T foliation. To carry out this calculation in practice, what we do is invert the transformation (105), obtaining ¹⁶

$$\begin{aligned}\tau &= (-a(x)/b(x)) + (1/b(x))T - (1/b(x))\delta T(x, T) \\ &= c(x) + h(x)T + \delta\tau(x, T),\end{aligned}\tag{109}$$

and then substitute both $\tau(x, T)$ and

$$d\tau = (\partial_k c + T\partial_k h + \partial_k \delta\tau)dx^k + (h + \frac{\partial(\delta\tau)}{\partial T})dT\tag{110}$$

into the expression (1) for $g(x, \tau)$, which for convenience we rewrite as

$$g(x, \tau) = -n^2(x, \tau)d\tau^2 + \gamma_{ij}dx^i dx^j.\tag{111}$$

Here the indices i, j, k run from 1 to 3; and $n(x, \tau)$ and $\gamma_{ij}(x, \tau)$ are certain combinations of e^ϕ , e^Λ , etc (see (1) along with the discussion just below it). We obtain

$$\begin{aligned}g(x, T) &= -n^2(h + \partial_T \delta\tau)^2 dT^2 - 2n^2(h + \partial_T \delta\tau)(\partial_i c + T\partial_i h + \partial_i \delta\tau)dT dx^i \\ &\quad + [\gamma_{ij} - n^2(\partial_i c + T\partial_i h + \partial_i \delta\tau)(\partial_j c + T\partial_j h + \partial_j \delta\tau)]dx^i dx^j\end{aligned}\tag{112}$$

¹⁶Our assumption that $b > 0$ and our verification that δT decays to zero for large τ guarantees that there exists $c(x) = (\frac{-a(x)}{b(x)})$, $h(x) = \frac{1}{b(x)} > 0$, and $\delta\tau(x, t) = (\frac{-1}{b(x)})\delta T$ with $\lim_{T \rightarrow \infty} \delta\tau = 0$ such that (109) is indeed the x -parametrized inverse to (105).

where $n(x, T) = n(x, c(x) + h(x)T + \delta\tau(x, T))$, etc. If we now compare expression (112) for g with the standard lapse-shift (“ADM”) form of a spacetime metric

$$g = -N^2 dt^2 + h_{ij}(dx^i + M^i dt)(dx^j + M^j dt), \quad (113)$$

then we find that the shift vector M can be expressed as

$$M^i = -\lambda^{ij}[n^2(h + \partial_T \delta\tau)(\partial_j c + T\partial_j h + \partial_j \delta\tau)] \quad (114)$$

where λ^{ij} is the inverse of the tensor appearing as the coefficients of $dx^i dx^j$ in (112).

To see the behavior of the shift M^i for large values of T , we need to write out n and λ^{ij} in terms of the variables Λ, ϕ, x, z , and then substitute into the asymptotic expressions for these quantities. In doing so, we again need to replace τ by its expression (109) in terms of T , noting that such quantities as $\delta\Lambda$ which decay for large τ do so as well for large T . We find that

$$M^i \approx e^{(2v_z - v_\Lambda)hT} \times (\text{polynomial in } T). \quad (115)$$

Thus, recalling our usual restrictions (54)-(56) on v_Λ and v_z , and also recalling that $h > 0$, we see that the shift decays rapidly to zero for large T . It is especially telling that the shift decays to zero much more quickly than does the lapse as one approaches the singularity, since we calculate

$$M^i M_i / N^2 \approx e^{(2v_z - v_\Lambda)hT} \times (\text{polynomial in } T). \quad (116)$$

So we conclude that, for any values of the harmonic time parameter functions $a(x)$ and $b(x) > 0$, the paths of the surface orthogonal observers corresponding to $T = a + b\tau + \delta T$ become increasingly parallel to those of the surface orthogonal observers corresponding to τ .

We now discuss how to verify that the observers which are orthogonal to the $T = a + b\tau + \delta T$ foliation see AVTD behavior in the chosen spacetime. The key first step is to find coordinates y^k which correspond to the T orthogonal observers. In practice, what we seek is a coordinate transformation

$$(x, T) \rightarrow (x(y, T), T) \quad (117)$$

which removes the shift term; ie, for which g has no $dydT$ term. A bit of calculation shows that if we choose the function $x(y, T)$ so that it satisfies

$$\partial_T x^i(y, T) = -M^i(x(y, T), T), \quad (118)$$

where M^i is the shift vector field discussed above, then indeed the metric written in terms of the coordinate (y, T) has vanishing shift, and so indeed the $y = \text{constant}$ observers are T surface compatible.

We note three features of the coordinate y generated via (118). First, the ODE system (118) together with a set of initial conditions $x^i(y, T_0) = \xi^i(y)$ constitute a well-posed system, with a unique local solution. Second, although the system (118) is generally nonlinear, it follows from the boundedness of M^i that (118) (together with the chosen initial conditions) determines a unique solution $x^i(y, T)$ for all T . Third, since M^i decays to zero exponentially rapidly for large T , we may write solutions $x^i(y, T)$ in the form

$$x^i(y, T) = X^i(y) + \delta x^i(y, T) \quad (119)$$

where $\delta x^i(y, T)$ decays to zero for large T .

With the transformation between x and y determined, we next rewrite the metric $g(x, t)$ in terms of the new coordinates. We are only interested in this for large T , so we may use expression (119), from which we derive

$$dx^i \approx X^i{}_{,j} dy^j - M^i dT. \quad (120)$$

Substituting this expression together with (119) into formula (112), and also replacing n and γ_{ij} by the appropriate expressions in terms of $\mathcal{A}_{pol} = \{\phi_o, \Lambda_o, x_o, z_o, v_\phi, v_\Lambda, v_x, v_z\}$ and T , we obtain a coordinate representation of the metric of the form

$$g(y, T) = g_{TT}(y, T) dT^2 + g_{ij}(y, T) dy^i dy^j, \quad (121)$$

with g_{TT} and g_{ij} as fairly complicated functions involving \mathcal{A}_{pol} along with the functions c , h , and $X^i(y)$.

To show that the T surface orthogonal observers see AVTD behavior, it is sufficient to verify the following:

Proposition 5 *For every choice of the asymptotic data \mathcal{A}_{pol} and for every choice of the harmonic time transformation functions $\{a, b > 0\}$, there exist functions $\{\tilde{\phi}(y), \tilde{\Lambda}_o(y), \tilde{x}_o(y), \tilde{z}_o(y), \tilde{v}_\phi(y), \tilde{v}_\Lambda(y), \tilde{v}_x(y), \tilde{v}_z(y)\}$, and functions $\{\delta\tilde{\phi}(y, T), \delta\tilde{\Lambda}(y, T), \delta\tilde{x}(y, T), \delta\tilde{z}(y, T), \delta\tilde{p}_\phi(y, T), \delta\tilde{p}_\Lambda(y, T), \delta\tilde{p}_x(y, T), \delta\tilde{p}_z(y, T)\}$ decaying to zero for large T , such that the metric coefficients g_{TT} and g_{ij} in (121) can be written in the polarized $U(1)$ Symmetric AVTD form described in sections 2 and 3; in particular, one has the asymptotic behaviors*

$$g_{TT} \approx -h^2 e^{2(\Lambda - 2\tau)} e^{-2\phi} \quad (122)$$

$$g_{ij} \approx \gamma_{mn} \frac{\partial X^m}{\partial y^i} \frac{\partial X^n}{\partial y^j} \quad (123)$$

The proof of this proposition is a straightforward (somewhat tedious) consequence of carrying through the details of calculation (121). We omit it here, together with the explicit (not especially useful) expressions one obtains for $\{\tilde{\phi}(y), \tilde{\Lambda}_o(y), \tilde{x}_o(y), \tilde{z}_o(y), \tilde{v}_\phi(y), \tilde{v}_\Lambda(y), \tilde{v}_x(y), \tilde{v}_z(y)\}$ in terms of \mathcal{A}_{pol} and $\{a, b > 0\}$.

Do we see AVTD behavior in these spacetimes using other foliations and sets of observers? This is not yet known. The foliations and observers we have discussed here exhibit two important features asymptotically: 1) All of the sets of surface compatible observers become parallel as one approaches the singularity. 2) In all of them, one sees AVTD behavior. It is not clear whether these features are related or not. It may be, for example, that in a spacetime which shows AVTD behavior with respect to one foliation and set of observers, AVTD behavior is seen by the surface compatible observers of any other foliation if and only if those observers become parallel asymptotically to the original ones. Or, it might be the case that these two features which coincide for our harmonic foliations and observers in these $U(1)$ Symmetric solutions are not closely related more generally. In either case, it may be that the surface compatible observers corresponding to most if not all spacelike foliations in a spacetime with AVTD behavior are asymptotically parallel, as one approaches the singularity. We hope to explore this issue in future work.

6 Generating Further $U(1)$ Symmetric Spacetimes with AVTD Behavior

Numerical work [7] suggests that general $U(1)$ Symmetric vacuum solutions on $T^3 \times R$ show Mixmaster behavior near the singularity. We may then ask if the $U(1)$ Symmetric solutions with AVTD behavior extends beyond the classes we have discussed here so far. In this section, we use Geroch's $SL(2, R)$ method of generating new $U(1)$ Symmetric solutions from old ones to show that this is the case.

We recall [19] [22] how the Geroch transformation works: If $\{\phi, \Lambda, w, x, z; -p_\phi, p_\Lambda, r, p_x, p_z\}$ is a solution of the $U(1)$ Symmetric vacuum Einstein equations on (Σ^3, R) , and if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a (constant) matrix contained in $SL(2, R)$

(so $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$), then $\{\hat{\phi}, \Lambda, \hat{w}, x, z; \hat{p}_\phi, p_\Lambda, \hat{r}, p_x \cdot p_z\}$ with

$$e^{2\hat{\phi}} = \frac{e^{2\phi}}{c^2(w^2 + e^{4\phi}) + 2cdw + d^2} \quad (124)$$

$$\hat{w} = \frac{ac(w^2 + e^{4\phi}) + (ad + bc)w + bd}{c^2(w^2 + e^{4\phi}) + 2cdw + d^2} \quad (125)$$

$$\hat{p}_\phi = \frac{p(c^2(w^2 - e^{4\phi}) + 2cdw + d^2) - r(4e^{4\phi}(cd + wc^2))}{c^2(w^2 + e^{4\phi}) + 2cdw + d^2} \quad (126)$$

$$\hat{r} = p(c^2w + cd) + r[d^2 + c^2(w^2 - e^{4\phi}) + 2cdw] \quad (127)$$

is also a solution of the $U(1)$ Symmetric vacuum Einstein equations on $\hat{\Sigma}^3 \times R$, where $\hat{\Sigma}^3$ is diffeomorphic to Σ^3 if and only if $\int_{\hat{\Sigma}^3} \hat{r} = \int_{\Sigma^3} r$.

We wish to consider Geroch transformations which map solutions on $\Sigma^3 = T^3$ to others also on T^3 . Since $\int_{\Sigma^3} r = 0$ if and only if $\Sigma^3 = T^3$, we seek solutions and transformations such that $\int_{\Sigma^3} r = 0$ both before and after the transformation. Using the three constants of the motion

$$A = \int_{\Sigma^3} (2wr + p_\phi), \quad (128)$$

$$B = \int_{\Sigma^3} r, \quad (129)$$

and

$$C = \int_{\Sigma^3} [r(e^{4\phi} - w^2) - p_\phi w], \quad (130)$$

one finds that if $\{A, B, C\}$ characterize the original solution, then

$$\hat{A} = (ad + bc)A + 2bdB - 2acC, \quad (131)$$

$$\hat{B} = d^2B - c^2C + cdA, \quad (132)$$

and

$$\hat{C} = a^2 C - b^2 B - abA \quad (133)$$

characterize the transformed solution. So we seek solutions and $SL(2, R)$ matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $B = 0$ and $d^2 B - c^2 C + cdA = 0$.

For polarized solutions, $r = 0 = w$ and $p_\phi \neq 0$, so $B = 0$, $C = 0$, and $A \neq 0$. Thus, to obtain $\hat{B} = 0$, we need to choose $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with either $c = 0$ or $d = 0$ (we can not have both zero, since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R)$). For half-polarized solutions as well, the asymptotic decay of w and $e^{4\phi}$ require that $C = 0$, so again we obtain $\hat{B} = 0$ if and only if either $c = 0$ or $d = 0$.

Applying an $SL(2, R)$ Geroch transformation with $c = 0$ to either a polarized or half-polarized solution produces nothing really new.¹⁷ The $d = 0$ transformation, however, produces geometrically new AVTD solutions. Applied to a polarized solution, such a transformation takes the form

$$e^{2\hat{\phi}} = (1/c^2)e^{-2\phi} \quad (134)$$

$$\hat{w} = a/c, \quad (135)$$

$$\hat{p}_\phi = -p_\phi \quad (136)$$

and

$$\hat{r} = 0. \quad (137)$$

The new solutions are geometrically distinct from the original ones, since $\int e^{2\phi} dx^3$ is the diameter of the (evolving) three-geometry along the $U(1)$ symmetry direction; and while this diameter decays to zero in the original solution, it blows up in the transformed solution. Note that one readily

¹⁷Such a transformation results in a (constant) rescaling of some of the quantities, and the addition of a constant to w . We recall that w does not appear in the metric, and it appears in the Hamiltonian and in the field equations only as a gradient term ∇w .

verifies that so long as the original solutions are AVTD, the transformed one are as well.

For the half-polarized solutions, the transformation corresponding to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ takes the form (with " \rightarrow " indicating asymptotic values)

$$e^{2\hat{\phi}} = \frac{e^{2\phi}}{c^2(w^2 + e^{4\phi})} \rightarrow e^{-2\phi}/c^2 \quad (138)$$

$$\begin{aligned} \hat{w} &= \frac{ac(w^2 + e^{4\phi}) + bcw}{c^2(w^2 + e^{4\phi})} \\ &\rightarrow \left(\frac{a}{c}\right) + \left(\frac{b}{c}\right) \left(\frac{-\overset{\circ}{r}}{4v_\phi}\right) \end{aligned} \quad (139)$$

$$\begin{aligned} \hat{p}_\phi &= \frac{p_\phi(c^2(w^2 - e^{4\phi})) - r(4e^{4\phi}c^2w)}{c^2(w^2 + e^{4\phi})} \\ &\rightarrow -p_\phi \end{aligned} \quad (140)$$

$$\begin{aligned} \hat{r} &= p(c^2w) + rc^2(w^2 - e^{4\phi}) \\ &\rightarrow 0. \end{aligned} \quad (141)$$

Here again, the inversion of $e^{2\phi}$ indicates geometrically new AVTD solutions. As well, while the half-polarized solutions discussed above have w asymptotically vanishing and r approaching a general function on T^2 (see equations (57)-(58)), these new solutions have r asymptotically vanishing and w approaching a general function on T^2 . Examining the Hamiltonian (6)-(7), we see that this swapping of the asymptotic behavior of r and w is needed to avoid having H blow up as a consequence of the inversion of $e^{2\phi}$.

7 Conclusion

How prevalent is AVTD behavior in solutions of Einstein's equations? Since this behavior is so specialized, it is perhaps rather surprising that, as we show here, it is found in substantial classes of solutions with only one Killing field.

As noted above, numerical work [7] indicates that in generic cosmological spacetimes with one Killing field, one finds Mixmaster rather than AVTD behavior near the singularity. However, it is likely that we can extend the classes of solutions known to have AVTD behavior in at least two ways.

First, we should be able to remove the analytic condition which our theorems here require. Rendall has shown how to do this for the class of Gowdy spacetime: In his work with Kichenassamy [8], it is shown that certain classes of analytic Gowdy spacetimes have AVTD behavior; then in his later work [10], he shows that the same holds for C^∞ solutions. Work has begun which applies techniques similar to those used in [10] to show that AVTD behavior is found in classes of C^∞ polarized $U(1)$ Symmetric spacetimes.

Second, we hope to be able to show that all of the polarized $U(1)$ Symmetric vacuum solutions have AVTD behavior, and not just a subset of these spacetimes. Numerical studies suggest that this is true; we are looking for ways to prove this contention.

It is not clear whether there are vacuum solutions with *no* Killing fields which exhibit AVTD behavior. Andersson and Rendall [11] have shown that there are classes of solutions of the Einstein equations coupled to a scalar field with AVTD behavior; however, while heuristic analyses have predicted this result, the same sorts of studies suggest that generic vacuum solutions with no Killing fields should show Mixmaster rather than AVTD behavior. We note that, to date, apart from the work of Berger and Moncrief [23] which uses Geroch transformations to generate a small class of $U(1)$ Symmetric spacetimes with Mixmaster behavior, there are very few rigorous results concerning the existence of such behavior in spatially inhomogeneous cosmological spacetimes.

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